

# Comment on “Operator Quantum Error Correction”

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The attempt to equate operator quantum error correction (quant-ph/0504189v1) with the quantum computer condition (quant-ph/0507141) in version two of quant-ph/0504189 is shown to be invalid.

After the appearance of our paper [1], the authors of [2] released a revised version (*cf* [3]), in which they make the following incorrect assertions about our paper:

**Assertion 1:** Equation (8) in our paper [1] is “the fundamental formula in the formulation of the ‘Quantum Computer Condition’ ” (*cf* the second sentence in the third paragraph of [3]);

**Assertion 2:** Equation (8) in our paper [1] is “captured as a special case of the UNS framework” (*cf* again the second sentence in the third paragraph of [3]).

In this Comment we demonstrate that the above assertions are erroneous.

**Response to 1:** Equation (8) in our paper [1] is

$$\mathcal{M}_{\text{dec}}(P \cdot (\mathcal{M}_{\text{enc}}(\rho))) = U \rho U^\dagger, \quad (1)$$

and is explicitly referred to by us in [1] as the “encoded *ersatz* quantum computer condition,” (eEQCC) with the understanding that the word *ersatz* would properly be understood as meaning *inferior* or *false*. Our paper clearly shows that eq.(8) of [1] is not fundamental in any respect. Indeed, eq.(8) of [1] was specifically and intentionally presented for the sole purpose of illustrating an *unacceptable* dynamical condition. This is explicitly pointed out in several places in our paper, such as in the following sentence (*cf* the first sentence in the second paragraph above Section 2.3 of [1]): “However, eq.(8) does *not* (emphasis added) in general provide an acceptable condition to connect the dynamics of a practical quantum computing device to the constraints implied by the unitary operator  $U$  that defines the abstract quantum computation.” In fact, the various results that were presented by us in [1] do not and indeed *cannot* follow from the encoded *ersatz* quantum computer condition. This is because the eEQCC lacks any quantification of implementation inaccuracy, *i.e.*, residual errors not fully removed by error correction. Rather, the results of our paper [1] follow from a different mathematical expression, the *proper* quantum computer condition (QCC), in which a parameter,  $\alpha$ , quantifies the implementation inaccuracy. This is discussed further in the second Remark section below.  $\square$

**Remark:** Claims of equivalence between the UNS of [2] and the eEQCC of [1] are irrelevant. None of the results of [1] could possibly flow from [2] since any suggested equivalence between the papers stems from the misidentification in [3] of the fundamental expression of [1]. Nevertheless, even the claimed identification of UNS with the eEQCC is wrong as we now show.  $\square$

**Response to 2:** The focus of UNS in [2] is the *error operators* of a quantum communications *channel*. This is evident, for example, in the proof of Theorem 6.1 in [2], where the Kraus operators associated to the product of  $U$  and  $\mathcal{E}$  are specifically referred to as “noise operators.” Note also the many places in [2] where  $\mathcal{E}$  is referred to as the “channel.” In complete contrast, the eEQCC of [1] is constructed from operators that describe the *global evolution* of a quantum *computer*. Any attempt to arbitrarily promote the error operators appearing in [2] to global evolution operators (note that global evolution includes errors as a special case, but not *vice versa*) would completely change the original meaning and scope of the UNS expression [4]. (The attempt made in [3] to replace an error operator with a global evolution operator is described in the next paragraph.) This could only be regarded as an attempt after-the-fact to emulate the physical content of the eEQCC.

Apart from the above, the claimed reduction outlined in [3] is not mathematically valid. According to [3]  $P$  acts on the computational Hilbert space,  $\mathcal{H}_{\text{comp}}$  (*cf* the sentence following eq.(19) of [3]: “... a Hilbert space  $\mathcal{H}_{\text{comp}}$ , on which  $P$  is a quantum operation.”). The authors of [3] introduce the composition  $\mathcal{E} = \mathcal{M}_{\text{dec}} \circ P \circ (\mathcal{M}_{\text{enc}} \oplus \text{id}_{\mathcal{K}})$  to effect their claimed reduction (*cf* the last sentence in the second paragraph below eq.(19) in [3]). However, in this composition the operator  $P$  acts on the Hilbert space  $\mathcal{H}_{\text{comp}} \oplus \mathcal{K}$ , since  $\mathcal{M}_{\text{enc}} : \mathcal{B}(\mathcal{H}_{\text{logical}}) \rightarrow \mathcal{B}(\mathcal{H}_{\text{comp}})$  (*cf* the last sentence in the paragraph containing eq.(19) in [3]). This makes the above composition inconsistently defined.

In the converse direction, the authors of [3] give the putative instantiation  $\mathcal{M}_{\text{enc}}(\sigma^B) = \mathbb{1}^A \otimes \sigma^B$  (*cf* the last sentence in the last paragraph of Section 4 of [3]). However, the actual presentation in our paper [1] explicitly refers to  $\mathcal{M}_{\text{enc}}$  and  $\mathcal{M}_{\text{dec}}$  as (*cf* second sentence before eq. (12) in [1]) “completely-positive, trace-preserving encoding

and decoding maps (with *no further restrictions of any kind*).” (Italics have been added here to the original text of [1].) The necessity in the claimed instantiation given in [3] to restrict operators  $\mathcal{M}_{\text{enc}}$  to a particular structure shows that the UNS equation is actually a special case of the eEQCC.

It should come as no surprise that the UNS equation is a special case of the eEQCC. After all, the eEQCC describes a quantum *computer* with perfect error correction, while the UNS equation describes a particular type of quantum *channel* with perfect error recovery, and a quantum channel is effectively a quantum computer that is intended to implement the identity operation.

In fact, the reduction of the eEQCC to the UNS equation can be shown explicitly by reference to Section 2.3.3 of our paper [1], where we demonstrated that the correctability criterion used in OQEC (*cf* [2]) is a special case of the *proper* QCC given in eq.(12) of [1]. We did this by first setting our parameter  $\alpha$  to zero, which has the effect, through the removal of a norm, of reducing the QCC to the eEQCC. We then showed explicitly how to specialize the operators appearing in the eEQCC so as to obtain the correctability criterion of OQEC [5]. With this result from [1], we now take note of the statement (*cf* the first full sentence following eq.(18) in [3]) that the UNS equation “is a special case of the OQEC formulation Eq. (13) where the recovery  $\mathcal{R}$  is unitary” [6]. Thus we see that the eEQCC is a special case of the QCC, the OQEC correctability criterion is a special case of the eEQCC, and (from the above-cited sentence in [3]) the UNS equation is a special case of OQEC correctability. Taken together, this establishes the fact that the UNS condition is a special case of the eEQCC.

We re-emphasize that, although Assertion 2 is in fact erroneous, it would be irrelevant to any of the results of our paper even if it were true: the mathematical condition from which the results in our paper *actually* derive

is the *proper* QCC given in eq.(12) of [1], as opposed to the clearly labeled encoded *ersatz* QCC in eq.(8) of [1].  $\square$

**Remark:** As a final remark, we note that the generalization of the eEQCC (eq.(8) of [1]) to the *proper* QCC (eq.(12) of [1]) involves several nontrivial issues, including subtle considerations pertaining to the proper choice of norm. An analysis of some of these issues is given in [7].  $\square$

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- [1] G. Gilbert, M. Hamrick and F. J. Thayer. A theory of physical quantum computation. *arXiv*: quant-ph/0507141. (Although *arXiv*: quant-ph/0507141 was written by only three of the authors (GG, MH and JT) of the present Comment, we will for brevity here refer to *arXiv*: quant-ph/0507141 as “our paper.”)
  - [2] D. Kribs, R. LaFlamme, D. Poulin, and M. Lesosky. Operator quantum error correction. *arXiv*: quant-ph/0504189 v1.
  - [3] D. Kribs, R. LaFlamme, D. Poulin, and M. Lesosky. Operator quantum error correction. *arXiv*: quant-ph/0504189 v2.
  - [4] Putative claims that  $\mathcal{E}$  actually describes the *global evolution* of a quantum computer would obviate the need to introduce any  $U$  at all. Though [2] implies (*cf* second sentence of section 2.1) that  $\mathcal{E}$  can be a general quantum evolution, careful inspection of [2] reveals that  $\mathcal{E}$  is actually used only as a noise operator for a quantum channel.
  - [5] In contrast to the presentation of the first claimed reduction given in [3], the analysis presented in [1] respects the semantic content of both the QCC and OQEC.
  - [6] Note that Eq. (13) in [3] is the OQEC correctability criterion of [2].
  - [7] G. Gilbert, M. Hamrick, F. J. Thayer and Y. S. Weinstein. Quantum Computer Condition: Stability, Norms and Classical Probabilistic Computation. In preparation.